Nonlinear Fokker–Planck equations as gradient flows on the space of probability measures

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For $m \geq 1$, consider Porous Media Equation on $\mathbb{R}^d$, $d \geq 1$

$$\partial_t u = \Delta(|u|^{m-1}u), \; u(0,\cdot) = \mu_0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{PME}$$

For every $\mu_0 \in \mathscr{P}$ = probability measures on $\mathbb{R}^d$, there is a distributional probability solution

$$u : t \mapsto u(t), \quad u(t,x)dx \longrightarrow \mu_0 \text{ as } t \to 0,$$

i.e. $u(t,x)dx \in \mathscr{P}$ for all t.

Our question: Is $t \mapsto u(t,x)dx$ a *gradient flow* in the "manifold" $\mathscr{P}$?

Previously: [Otto01] showed a formal gradient flow structure in the measure metric space $\mathscr{P}_2$.

Our answer: Yes, with a more rigorous, purely geometric approach, applicable to larger class of *generalized PME*

$$\partial_t u = \Delta\beta(u) - \operatorname{div}(D(x)b(u)u),$$

where $\beta : \mathbb{R} \to \mathbb{R}$ monotone, $D : \mathbb{R}^d \to \mathbb{R}^d$, $b : \mathbb{R} \to \mathbb{R}$.

Geometry and gradient flows on $\mathscr{P}$

Gradient flows

For a Riemannian manifold M with gradient ∇^M and $E : M \to \mathbb{R}$, a *gradient flow (wrt. E)* is the equation for $x : t \mapsto M$

$$\dot{x}_t = -\nabla^M E(x_t), \quad t \geq 0,$$

to be solved in the tangent bundle of M.

Hence: First need a Riemannian differential geometry on $\mathscr{P}$. It can be obtained by "lifting the geometry from $\mathbb{R}^d$ to $\mathscr{P}$":

Riemannian geometry on $\mathscr{P}$

Test functions = finitely-based functions

$$\mathscr{F}C_b^2 := \{F : \mathscr{P} \to \mathbb{R} \,|\, F(\mu) = f(\mu(h_1), \ldots, \mu(h_n)), n \in \mathbb{N}, h_i \in C_c^2, f \in C_b^1(\mathbb{R}^n)\}$$

where $\mu(h) = \int_{\mathbb{R}^d} h \, d\mu$.

To obtain a tangent space $T_\nu \mathscr{P}$ at $\nu \in \mathscr{P}$, consider for $\psi \in L^2(\mathbb{R}^d, \mathbb{R}; \nu)$

$$t \mapsto \mu_t^{\nu,\psi} := \nu \circ (\mathrm{Id} + t\psi)^{-1}.$$

Since for $F \in \mathscr{F}C_b^2$, $F : \mu \mapsto f(\mu(h_1), \dots, \mu(h_n))$, one has

$$\mathrm{diff}_\nu F(\psi) = \frac{d}{dt} F(\mu_t^{\nu,\psi})_{|t=0} = \Big\langle \sum_{k=1}^{n} \partial_k f(\nu(h_1), \dots, \nu(h_n)) \nabla h_k, \psi \Big\rangle_{L^2(\nu)},$$

one is led to consider

$$T_\nu \mathscr{P} := L^2(\mathbb{R}^d, \mathbb{R}^d; \nu)$$

and

$$\nabla^{\mathscr{P}} F(\nu) := \sum_{k=1}^{n} \partial_k f(\nu(h_1), \dots, \nu(h_n)) \nabla h_k.$$

This geometry is not an arbitrary one, but the one obtained by the aforementioned "geometry lifting" from $\mathbb{R}^d$ to $\mathscr{P}$. Also, $\nabla^{\mathscr{P}} E$ can be extended to larger classes of functions $E : \mathscr{P} \to \mathbb{R}$.

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The Porous Media Equation

Let $\mu_0 \in \mathscr{P}$ and let $u : t \mapsto u(t)$ be the unique probability solution to (PME) with initial datum μ_0 in $\cap_{\delta>0} L^\infty((\delta,\infty)\times\mathbb{R}^d)$.

Theorem (R./Röckner23)

$\mu_t := u(t,x)dx$ solves the gradient flow

$$\frac{d}{dt}\mu_t = -\nabla^{\mathscr{P}} E(\mu_t), \quad t \geq 0$$

in $\mathscr{P}$, with energy

$$E : \mu = u(x)dx \mapsto \frac{1}{m-1}\int_{\mathbb{R}^d} u(x)^m \, dx,$$

and $\nabla^{\mathscr{P}} E(\mu_t) = \frac{\nabla(u(t)^m)}{u(t)} \in L^2(\mathbb{R}^d,\mathbb{R}^d;\mu_t)$. If $d \geq 3$ and $\mu_0 \in L^\infty$, then

$$\int_0^T |\nabla^{\mathscr{P}} E(\mu_t)|^2_{T_{\mu_t}\mathscr{P}} \, dt < \infty, \quad \forall T > 0.$$

Generalized Porous Media Equation

To treat

$$\partial_t u = \Delta\beta(u) - \operatorname{div}\big(D(x)b(u)u\big), \qquad \text{(gPME)}$$

change the geometry on $\mathscr{P}$: Instead of

$$\big(T_\nu\mathscr{P}, \langle\cdot,\cdot\rangle_{T_\nu\mathscr{P}}\big) = \big(L^2(\mathbb{R}^d,\mathbb{R}^d), \langle\cdot,\cdot\rangle_{L^2(\nu)}\big),$$

consider weighted metric tensor:

$$\big(T_\nu\mathscr{P}, \langle\cdot,\cdot\rangle_{T_\nu\mathscr{P}}\big) = \big(L^2(\mathbb{R}^d,\mathbb{R}^d), \langle b(u)^{-1}\cdot,\cdot\rangle_{L^2(\nu)}\big),$$

which leads to a different gradient $\nabla_b^{\mathscr{P}}$.

Assumptions on β, D, b

(i) $\beta \in C^1(\mathbb{R}),\ \beta(0) = 0,\ \gamma \le \beta'(r) \le \gamma_1,\ r \in \mathbb{R}$, for $0 < \gamma < \gamma_1 < \infty$.

(ii) $b \in C_b(\mathbb{R}) \cap C^1(\mathbb{R}), b \ge b_0 > 0$.

(iii) $\Phi \in C^1(\mathbb{R}^d),\ \nabla\Phi \in C_b(\mathbb{R}^d, \mathbb{R}^d),\ D = -\nabla\Phi$.

(iv) $(\mathrm{div}\, D)^- \in L^\infty(\mathbb{R}^d)$ and $(\mathrm{div}\, D)^+ \in (L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))$.

(v) $\Phi \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d),\ \Phi \ge 1,\ \lim\limits_{|x|\to\infty} \Phi(x) = +\infty$ and there exists $m \in [2, \infty)$ such that $\Phi^{-m} \in L^1(\mathbb{R}^d)$.

Let $\eta(r) := \int_0^r \int_1^s \frac{\beta'(w)}{wb(w)} dw\, ds$, $r \in \mathbb{R}_+$, and

$$E(v(x)dx) := \int_{\mathbb{R}^d} \eta(v(x))\, dx + \int_{\mathbb{R}^d} \Phi(x)\, v(x) dx.$$

Let $\mu_0 = v_0(x)dx, v_0 \in L^\infty(\mathbb{R}^d)$ such that $v_0 \log v_0 \in L^1(\mathbb{R}^d)$.

Theorem (R./Röckner23)

There is a unique solution $t \mapsto \mu_t = u(t,x)dx$ to (gPME) in $L^\infty([0,\infty] \times \mathbb{R}^d)$, and it satisfies the gradient flow

$$\frac{d}{dt}\mu_t = -\nabla_b^{\mathscr{P}} E(\mu_t), \quad t \geq 0$$

in $\mathscr{P}$, where E is as above. Moreover,

$$\nabla_b^{\mathscr{P}} E(\mu_t) = \frac{\nabla(u(t)^m)}{u(t)} - b(u(t))D \in L^2(\mathbb{R}^d, \mathbb{R}^d; \mu_t).$$

Further results and remarks

(i) If, in addition, the "balance condition"

$$\gamma_1 \Delta\Phi(x) - b_0 |\nabla\Phi(x)|^2 \leq 0, \quad dx - \text{a.s.},$$

holds, the energy E is a Lyapunov function for $t \mapsto \mu_t$, i.e. $E(\mu_t) \leq E(\mu_s), s \leq t$. Moreover, $u(t) \longrightarrow u_\infty$ in $L^1(\mathbb{R}^d)$ as $t \to \infty$, and u_∞ can explicitly be calculated from E.

(ii) For $\beta(r) = \sigma r$, $\sigma \in (0,\infty)$, $b(r) = b_0 \in (0,\infty)$, E is the classical Boltzmann entropy function.

(iii) $\nabla_b^{\mathscr{P}} E(\mu_t) \in \overline{\{b(u(t))\nabla\zeta \,|\, \zeta \in C_c^\infty(\mathbb{R}^d)\}}$, where the closure is taken in

$$\left(L^2(\mathbb{R}^d,\mathbb{R}^d;\mu_t), \langle b(u(t))^{-1}\cdot,\cdot\rangle_{L^2(\mu_t)}\right).$$

(iv) Extensions of our results to more general equations

$$\partial_t u = \operatorname{div}\left(A(u)\nabla u\right) - \operatorname{div}\left(B(u)D(x)u\right),$$

where $A, B : \mathbb{R} \to \mathbb{R}^{d\times d}$ are matrix-valued seem possible.

Thank you for your attention!