On nonlinear Markov processes in the sense of McKean

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Table of Contents

FPEs, SDEs, Markov processes: Linear case

For $a = (a_{ij})_{i,j \leq d} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, b = (b_i)_{i \leq d} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, consider linear Fokker–Planck equation (FPE)

$$\partial_t \mu_t = \partial^2_{ij}(a_{ij}(t,x)\mu_t) - \partial_i(b_i(t,x)\mu_t), \quad t \geq s, \quad \mu_s = \zeta \in \mathscr{P} \qquad (\ell\text{FP})$$

in weak sense, a 2nd-order parabolic PDE for measures $t \mapsto \mu_t \in \mathscr{P}$.

Corresponding SDE:

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad t \geq s, \quad X_s \sim \zeta, \qquad (\text{SDE})$$

$B = \mathbb{R}^d$-Brownian motion, $\frac{1}{2}\sigma\sigma^T = a$.

Connection between (ℓFP) and (SDE):

- X weak solution to (SDE) $\implies$ $(\mu_t)_{t \geq s} := (\mathscr{L}_{X_t})_{t \geq s}$ solves (ℓFP).
- $(\mu_t)_{t \geq s}$ solution to (ℓFP) $\implies$ $\exists$ weak solution X to (SDE) with

$$\mathscr{L}_{X_t} = \mu_t.$$

FPEs, SDEs, Markov processes: Linear case

Notation: $\Omega_s := C([s,\infty),\mathbb{R}^d)$, $\pi_t^s : \Omega_s \to \mathbb{R}^d$, $\pi_t^s(w) := w(t)$, $\mathscr{F}_{s,r} := \sigma(\pi_\tau^s, s \leq \tau \leq r)$, "time-inhomogeneous canonical model".

Classical result:
$\exists!$ solution $(\mu_t^{s,\zeta})_{t\geq s}$ to (ℓFP) from each i.d. $(s,\zeta) \in \mathbb{R}_+ \times \mathscr{P}$
$\implies$ (SDE) weakly well-posed
$\implies$ Unique solution laws $\mathbb{P}_{s,\zeta}$ on Ω_s have marginals $\mu_t^{s,\zeta}$ and form a linear Markov process, i.e. for $\mathbb{P}_{s,x} := \mathbb{P}_{s,\delta_x}$, $s \leq r \leq t$:

$$\mathbb{P}_{s,x}(\pi_t^s \in A | \mathscr{F}_{s,r})(\cdot) = \mathbb{P}_{r,\pi_r^s(\cdot)}(\pi_t^r \in A) \quad \mathbb{P}_{s,x}\text{-a.s.}, \forall A \in \mathscr{B}(\mathbb{R}^d)$$

and $\mathbb{P}_{s,\zeta} = \int_{\mathbb{R}^d} \mathbb{P}_{s,y}\, d\zeta(y)$ for all $\zeta \in \mathscr{P}$.

This Markov process is uniquely determined by its marginals, so we have a **one-to-one correspondence** between Markov processes and well-posed (ℓFP)-equations.

FPEs, SDEs, Markov processes: Linear case

Central linear example: Brownian motion and heat equation:
For $b = 0$, $a = \mathrm{Id}$: (ℓFP) = heat equation, with corresponding SDE

$$dX_t = dB_t, \quad t \geq s, \quad X_s \sim \zeta,$$

and the path laws of its unique solutions, i.e. the translated Wiener measures, form a linear Markov process.

Our motivation and goals

Question: Similar connection between nonlinear FPEs, distribution-dependent SDEs and Markov processes?

Precisely: for $a_{ij}, b_i : \mathbb{R}_+ \times \mathscr{P} \times \mathbb{R}^d \to \mathbb{R}$, we want to

(i) start with solutions $(\mu_t^{s,\zeta})_{t \geq s}$ to nonlinear FPE

$$\partial_t \mu_t = \partial_{ij}^2 (a_{ij}(t, \mu_t, x)\mu_t) - \partial_i (b_i(t, \mu_t, x)\mu_t), \quad t \geq s, \quad \mu_s = \zeta, \quad (n\ell\text{FP})$$

(ii) lift to path laws $\mathbb{P}_{s,\zeta}$ of weak solutions to distribution-dependent SDE

$$dX_t = b(t, \mathscr{L}_{X_t}, X_t)dt + \sigma(t, \mathscr{L}_{X_t}, X_t)dB_t, \quad t \geq s, \quad X_s \sim \zeta, \quad (\text{DDSDE})$$

(iii) and prove $\{\mathbb{P}_{s,\zeta}\}_{s,\zeta}$ is Markov.

Note: (ii) ✓, by nonlinear extension of Ambrosio-Figalli-Trevisan superposition principle in [Barbu/Röckner18].

Special case: Nemytskii-type FPEs = PDEs

Important class of nonlinear FPEs: *Nemytskii coefficients*

$$a(t,\mu,x) = \tilde{a}\Big(t, \frac{d\mu}{dx}(x), x\Big),\ b(t,\mu,x) = \tilde{b}\Big(t, \frac{d\mu}{dx}(x), x\Big).$$

For such coefficients:

- ($n\ell$FP) as equation for densities $t \mapsto u_t = \frac{d\mu_t}{dx}$ reads

$$\partial_t u_t = \partial^2_{ij}(a_{ij}(t,u_t,x)u_t) - \partial_i(b_i(t,u_t,x)u_t), \quad t \geq s, \quad u_t(x)dx \xrightarrow{t\to s} \zeta.$$

- $\mu \mapsto a(t,\mu,x), b(t,\mu,x)$ NOT continuous wrt. weak topology on $\mathscr{P}$.

Examples: Porous media, Burgers, $2D$ vorticity Navier–Stokes,...see later.

We want to include such equations in our theory.

Difficulties?

- Nonlinear FPEs are usually not well-posed.
- Even *if* ($n\ell$FP) well-posed: Classical Markov property not satisfied, since solutions not stable w.r.t. linear combinations in initial datum.

$\Longrightarrow$ New reasonable notion of "nonlinear" Markov processes required.

By *reasonable* we mean:

- "*Future is independent of the past given the present*".
- Path laws are uniquely determined by a family of kernels.
- Marginals satisfy a dynamical equation (linear case: Chapman-Kolmogorov).

Why?

- Connection between FPEs/PDEs and Markov processes is powerful in linear case: *probabilistic representation of PDEs as marginals of Markov processes.*
- In 1966 H.P.McKean suggested to generalize the Markov property such that it applies to processes with marginals given by solutions to nonlinear PDEs.
  But at his time: Limited theory of ($n\ell$FP) and (DDSDE), thus such a program was not developed.

Our aim: Definition, theory, applications of nonlinear Markov processes.

Table of Contents

Nonlinear Markov processes: Definition

Definition

Let $\mathscr{P}_0 \subseteq \mathscr{P}$ (= allowed initial data). A family $\{\mathbb{P}_{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathscr{P}_0}$ of path measures is a *nonlinear Markov process*, if for all $0 \leq s \leq r \leq t$:

(i) $\mathbb{P}_{s,\zeta} \circ (\pi^s_t)^{-1} =: \mu^{s,\zeta}_t \in \mathscr{P}_0$.

(ii) The *nonlinear Markov property* holds: $\forall A \in \mathscr{B}(\mathbb{R}^d)$

$$\mathbb{P}_{s,\zeta}(\pi^s_t \in A|\mathscr{F}_{s,r})(\cdot) = \mathbb{P}_{r,\mu^{s,\zeta}_r}(\pi^r_t \in A|\pi^r_r = \pi^r_r(\cdot))\ \mathbb{P}_{s,\zeta}-\text{a.s.} \quad (n\ell\text{MP})$$

Sanity checks:

- Linear Markov processes are special cases, with $\mathscr{P}_0 = \mathscr{P}$ and

$$\mathbb{P}_{r,\mu^{s,\zeta}_r}(\cdot\,|\pi^r_r = y) = \mathbb{P}_{r,\delta_y} \quad \checkmark$$

Sanity checks

- Since $\mathbb{P}_{r,\mu_r^{s,\zeta}}(\pi_t^r \in A | \pi_r^r = \pi_r^r(\cdot))$ is a function of $(r, \pi_r^r, \mu_r^{s,\zeta})$:

  "Future is independent of past given the present" ✓

- Straightforward calculation: for Borel $f : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$

$$\mathbb{E}_{s,\zeta}[f(\pi_{t_0}^s, \dots, \pi_{t_n}^s)]$$
$$= \int_{\mathbb{R}^d} \left(\cdots \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x_0 \dots, x_n)\, p^{s,\zeta}_{t_{n-1},t_n}(x_{n-1}, dx_n) \right) p^{s,\zeta}_{t_{n-2},t_{n-1}}(x_{n-2}, dx_{n-1}) \dots \right) \mu_{t_0}^{s,\zeta}(dx_0),$$

$$p^{s,\zeta}_{t_i,t_{i+1}}(x, dz) := \mathbb{P}_{t_i,\mu_{t_i}^{s,\zeta}}(dz | \pi_{t_i}^{t_i} = x) \circ (\pi_{t_{i+1}}^{t_i})^{-1},$$

  i.e. $\mathbb{P}_{s,\zeta}$ uniquely determined by a family of kernels ✓

- *Flow property:* $\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}$ ✓

Flow property vs. Chapman-Kolmogorov

Recall: Marginals of linear Markov processes satisfy *Chapman-Kolmogorov equations*

$$\mu_t^{s,x} = \int_{\mathbb{R}^d} \mu_t^{r,y} \, d\mu_r^{s,x}(y), \quad \forall 0 \leq s \leq r \leq t, x \in \mathbb{R}^d,$$

by linearity of $\zeta \mapsto \mathbb{P}_{s,\zeta}$. In nonlinear case: Not satisfied.

We will see: The flow property is the key property for marginals of nonlinear Markov processes.

Table of Contents

Main result: Preparation

For nonlinear FPE

$$\partial_t \mu_t = \partial_{ij}^2 (a(t,\mu_t,x)\mu_t) - \partial_i (b_i(t,\mu_t,x)\mu_t), \quad t \geq s, \quad \mu_s = \zeta, \quad (n\ell\text{FP})$$

linearize it: replace μ_t by fixed weakly continuous curve $t \mapsto \nu_t \in \mathscr{P}$:

$$\partial_t \mu_t = \partial_{ij}^2 (a_{ij}(t,\nu_t,x)\mu_t) - \partial_i (b_i(t,\nu_t,x)\mu_t), \quad t \geq s, \quad \mu_s = \zeta. \quad (\nu\text{-}\ell\text{FP})$$

Solution $t \mapsto \mu_t$ to ($n\ell$FP) also solves (μ-ℓFP) ("its own linearized eq.").

$\implies$ Solution family $\{\mu^{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathscr{P}_0}$ to ($n\ell$FP) has associated family of linearized FPEs ($\mu^{s,\zeta}$-ℓFP).

Notation: $M_\nu^{s,\zeta}$ = (convex) set of all solutions to (ν-ℓFP) with i.d. (s,ζ)
$M_{\nu,\text{ex}}^{s,\zeta}$ = set of extremal points of $M_\nu^{s,\zeta}$.

Main result: Construction of nonlinear Markov processes

Theorem (R./Röckner)

Let $\mathscr{P}_0 \subseteq \mathscr{P}$ *and* $\{\mu^{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathscr{P}_0}$ *be a solution flow to* (nℓFP) *such that* $\mu^{s,\zeta} \in M^{s,\zeta}_{\mu^{s,\zeta},ex}$.
Then the path laws $\{\mathbb{P}_{s,\zeta}\}_{s\in\mathbb{R}_+,\zeta\in\mathscr{P}_0}$ *of unique weak solutions* $X^{s,\zeta}$ *with marginals* $\mu_t^{s,\zeta}$ *to associated* (DDSDE) *form a uniquely determined nonlinear Markov process.*

- (nℓFP)-solutions $\mu^{s,\zeta}$ need not be unique, but only form a flow!
- Second assumption: extremality of each $\mu^{s,\zeta}$ in set of solutions of "its own" linearized equation ($\mu^{s,\zeta}$-ℓFP).
- The assertion contains an implicit uniqueness result for (DDSDE).

A new uniqueness result

The uniqueness claim is a new result itself:

Theorem (R./Röckner)

Let $\mathscr{P}_0 \subseteq \mathscr{P}$ *and* $\{\mu^{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathscr{P}_0}$ *be a solution flow to* ($n\ell$FP) *such that* $\mu^{s,\zeta} \in M^{s,\zeta}_{\mu^{s,\zeta},ex}$ $\forall 0 \leq s \leq t, \zeta \in \mathscr{P}_0$. *Then:*
For every $(s,\zeta) \in \mathbb{R}_+ \times \mathscr{P}_0$, *there is a* **unique** *weak solution* $X^{s,\zeta}$ *to the corresponding* (DDSDE) *with marginals* $(\mu^{s,\zeta}_t)_{t\geq s}$.

Existence of $X^{s,\zeta}$ follows by nonlinear superposition principle.
Uniqueness part: new.

A useful characterization of the extremality condition

Notation: For $\mu : [s,\infty) \ni t \mapsto \mu_t \in \mathscr{P}$, set

$$\mathscr{A}_{s,\leq}(\mu) := \{(\eta_t)_{t\geq s} \subseteq \mathscr{P} : \eta_t \leq C\mu_t \ \forall t \geq s \text{ for some } C > 0\}.$$

Let $\mu^{s,\zeta}$ solve ($n\ell$FP) with i.d. (s,ζ). The following new lemma is helpful for applications.

Lemma (R./Röckner22)

$$\#(M^{s,\zeta}_{\mu^{s,\zeta}} \cap \mathscr{A}_{s,\leq}(\mu^{s,\zeta})) = 1 \iff \mu^{s,\zeta} \in M^{s,\zeta}_{\mu^{s,\zeta},ex}.$$

Note: $\mu^{s,\zeta} \in M^{s,\zeta}_{\mu^{s,\zeta}} \cap \mathscr{A}_{s,\leq}(\mu^{s,\zeta})$.

Main results: Idea of proof

Crucial tool for the proof is the following lemma (new itself):

Lemma (R./Röckner)

Let $\mu : [s,\infty) \to \mathscr{P}$ be the unique solution to a linear FPE in $\mathscr{A}_{s,\leq}(\mu)$ with i.d. (s,μ_0). Then, in the same class, solutions are also unique from any initial datum $(s, g d\mu_0)$, where g is a probability density wrt. μ_0 which is bounded above and below away from 0.

Main results: Idea of proof

Goal of proof: Find path measures P_1, P_2 on Ω_r such that nonlinear Markov property is equivalent to

$$P_1 \circ (\pi_t^r)^{-1}(A) = P_2 \circ (\pi_t^r)^{-1}(A), \quad \forall t \geq r.$$

To prove the latter, it suffices to show

- $t \mapsto P_i \circ (\pi_t^r)^{-1}$ solves ($\mu^{s,\zeta}$-ℓFPE),
- $P_1 \circ (\pi_r^r)^{-1} = P_2 \circ (\pi_r^r)^{-1}$,
- $P_i \circ (\pi_t^r)^{-1} = g \, d\mu_t^{s,\zeta}$, $t \geq r$, with g as in the previous lemma.

Table of Contents

Main example: Porous media eq. and Barenblatt solutions

Consider the classical porous media equation with initial datum (s,ζ)

$$\partial_t u = \Delta(u^m), \quad (t,x) \in [s,\infty) \times \mathbb{R}^d, \quad u(t,x)dx \xrightarrow{t\to s} \zeta \in \mathscr{P}$$

$m \geq 1$, as a nonlinear FPE, i.e. for $\mu = u(x)dx \in \mathscr{P}$

$$a_{ij}(t,\mu,x) = \delta_{ij} u^{m-1}(x), \quad b_i = 0.$$

For $\zeta = \delta_{x_0}$, a special solution is the *Barenblatt solution*

$$u^{s,x_0}(t,x) = (t-s)^{-\alpha} \left[\left(C - k|x-x_0|^2 (t-s)^{-2\beta} \right)^+ \right]^{\frac{1}{m-1}}.$$

We prove:

- $\exists$ flow $\{u^{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathscr{P}}$ with $u^{s,\delta_x} = u^{s,x_0}$,
- $u^{s,\zeta}$ is the restricted-unique solution to ($u^{s,\zeta}$-ℓPME).

Hence...

Main example continued

...by our result: There is a nonlinear Markov process $\{\mathbb{P}_{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathscr{P}}$ of solution laws to

$$dX_t = \sqrt{2u(t,X_t)^{m-1}}dB_t, \quad \mathscr{L}_{X_t} = u(t)dx = u^{s,\zeta}(t)dx, \quad t \geq s, \quad \mathscr{L}_{X_s} = \zeta,$$

and with u^{s,δ_x} = Barenblatt solutions.

- This Markov process is uniquely determined by its marginals.
- In particular, we obtain a *probabilistic representation* of the Barenblatt solutions as the marginals of a nonlinear Markov process.

More examples

Further examples can be treated similarly:

- **Generalized PME:**

$$\partial_t u_t = \Delta\beta(u_t) - \mathrm{div}\left(Db_0(u_t)u_t\right), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

  with DDSDE

$$dX_t = b_0\left(u_t(X_t)\right)D(X_t))dt + \sqrt{\frac{2\beta(u_t(X_t))}{u_t(X_t)}}dB_t, \ \mathscr{L}_{X_t} = u_t\,dx$$

- **Burgers' equation:**

$$\partial_t u = \frac{\partial^2 u}{\partial^2 x} - u\frac{\partial u}{\partial x}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

  with DDSDE

$$dX_t = \frac{1}{2}u_t(X_t)dt + dB_t, \quad \mathscr{L}_{X_t} = u_t(x)\,dx.$$

More examples

- $2D$ **vorticity Navier–Stokes equation:**

$$\partial_t \omega = \Delta \omega - \operatorname{div}\left(v\omega\right), \quad v = K * \omega,$$

where K is the Biot–Savart kernel $K(x) = \frac{(-x_2, x_1)}{2\pi |x|^2}$, $x \in \mathbb{R}^2$, with DDSDE

$$dX_t = (K * \omega(t))(X_t)dt + \sqrt{2}dB_t, \quad \mathscr{L}_{X_t} = \omega(t,x)dx.$$

Table of Contents

Outlook to future work

Done: Definition, elementary properties, construction from flows to nonlinear FPEs, important examples.

Future plan: Develop a rich theory of nonlinear Markov processes, similar to the linear case:

- More basic theory (e.g. strong nonlinear Markov property)
- Generators, "semigroups", "nonlinear Feller property"?
- Ergodicity of nonlinear Markov processes
- General state spaces (SPDEs)
- ...

...it seems: there is a lot to do!

Thank you for your attention!