Weighted L^1-semigroup approach for nonlinear Fokker–Planck equations and generalized Ornstein–Uhlenbeck processes

Marco Rehmeier
Scuola Normale Superiore Pisa & Bielefeld University
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FPEs, SDEs, Markov processes: Linear case

For $a=(a_{ij})_{i,j\leq d}:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^{d\times d}, b=(b_i)_{i\leq d}:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ Borel measurable, consider linear Fokker–Planck equation (FPE)

$$\partial_t\mu_t=\partial^2_{ij}(a_{ij}(t,x)\mu_t)-\partial_i(b_i(t,x)\mu_t),\quad t\geq s,\quad \mu_s=\zeta\in\mathscr{P},\quad (\ell\text{FPE})$$

$\mathscr{P}$ = prob. measures on $\mathscr{B}(\mathbb{R}^d)$.

Corresponding SDE, with $\frac{1}{2}\sigma\sigma^T=a$, is

$$dX_t=b(t,X_t)dt+\sigma(t,X_t)dB_t,\quad t\geq s,\quad X_s\sim\zeta.\quad (\text{SDE})$$

Recall:

- X solution to (SDE) $\implies$ $(\mu_t)_{t\geq s}:=(\mathscr{L}_{X_t})_{t\geq s}$ solves (ℓFPE).
- $(\mu_t)_{t\geq s}$ solution to (ℓFPE) $\implies$ $\exists$ solution X to (SDE) with $\mathscr{L}_{X_t}=\mu_t$.
- If (ℓFPE) is well-posed, so is (SDE), and unique solution laws of the latter are a Markov process with one-dim. marginals equal to (ℓFPE)-solutions.

This connection to Markov processes fails, if the FPE is nonlinear, i.e.

$$\partial_t \mu_t = \partial_{ij}^2\big(a_{ij}(t,\mu_t,x)\mu_t\big) - \partial_i\big(b_i(t,\mu_t,x)\mu_t\big), \quad t \geq s, \quad \mu_s = \zeta, \ (\text{n}\ell\text{FPE})$$

since the corresponding stochastic equation is distribution-dependent:

$$dX_t = b(t,\mathscr{L}_{X_t},X_t)dt + \sigma(t,\mathscr{L}_{X_t},X_t)dB_t, \quad t \geq s, \quad X_s \sim \zeta, \quad (\text{DDSDE})$$

and even in well-posedness cases, solutions to DDSDEs do NOT satisfy the usual Markov property.

To build a theory as in the linear case, in [R./Röckner22] we introduced and studied a notion of *nonlinear Markov processes*, and proved:

Given a semigroup of solutions to a nlFPE, there is a nonlinear Markov process, consisting of solution laws to the DDSDE with one-dim. marginals given by the semigroup.

This gives a *probabilistic representation* of nlFPE-solutions as marginals of nonlinear Markov processes.
Note: We do NOT need uniqueness of the nlFPE.

Nemytskii-type FPEs = interesting class of PDEs

Consider local, singular dependence of coefficients on measure:

$$a(t,\mu,x) = \tilde{a}\left(t, \frac{d\mu}{dx}(x), x\right),\ b(t,\mu,x) = \tilde{b}\left(t, \frac{d\mu}{dx}(x), x\right),$$

for $\tilde{a} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $\tilde{b} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$.
Rewritting the FPE for densities $t \mapsto u_t = \frac{d\mu_t}{dx}$ gives a nonlinear PDE:

$$du_t = \partial^2_{ij}(a_{ij}(t,u_t,x)u_t) - \partial_i(b_i(t,u_t,x)u_t), \quad t \geq s, \quad u_t(x)dx \xrightarrow{t\to s} \zeta.$$

Interesting PDEs are of this type: e.g. Burgers, classical and generalized PME, 2D vorticity Navier–Stokes. For all of these:

There is a nonlinear Markov process with one-dim. marginals equal to solutions to this equation, see [R./Röckner22], [BarbuRöcknerZhang23].

Next goals:

- Develop theory of nonlinear Markov processes and apply it to nonlinear PDEs and DDSDE to prove new results.
- Solve new classes of nlFPE and link them to nonlinear Markov processes. **This we do today!**

Good method to construct solution semigroups for Nemytskii-type FPE: Crandall–Liggett nonlinear semigroup approach in $L^1(\mathbb{R}^d, \mathbb{R}; dx)$. This way, [BarbuRöckner21] solved

$$\partial_t u(t) = \Delta\beta(u(t)) - \operatorname{div}\left(D(x)b(u(t))u(t)\right)$$

for curves of probability densities $t \mapsto u(t)$. Then: Connection to DDSDE, nonlinear Markov processes...

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Our goal: Solve

$$\partial_t u = \Delta\beta(u) - \nabla\Phi\cdot\nabla\beta(u) - \mathrm{div}_\rho(D(x)b(u)u) \qquad (\mathrm{FP}_\rho)$$

by semigroup method in $L^1(\mathbb{R}^d;\rho\,dx)$, where

$$\rho:\mathbb{R}^d\to\mathbb{R}\ L^1\text{-(probability) density, } \Phi=-\log\rho \text{ (i.e. } \rho=e^{-\Phi}),$$

div_ρ = dual of ∇ in $L^2(\mathbb{R}^d;\rho\,dx)$.

Example: $\rho(x)=e^{-\frac{|x|^2}{2}}, \Phi=\frac{|x|^2}{2}$ (Ornstein–Uhlenbeck processes, see later).

Theorem (R.23)

Assume the following hypothesis.

(H1) $0 \leq \Phi \in C^2$ *is convex with* $\lim_{|x|\to\infty} \Phi(x) = \infty$, *and* $\nabla\Phi \in L^1(\rho)$.

(H2) $\beta \in C^2(\mathbb{R})$, $\beta'(r) > 0$ *for* $r \neq 0$, *and* $\beta(0) = 0$.

(H3) $b \geq 0$, $b \in C_b(\mathbb{R})$.

(H4) $D \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $div_\rho D \in L^2_{loc}$, $(div_\rho D)^- \in L^\infty$.

Then, for each $u_0 \in L^1(\rho\, dx) \cap L^\infty$, *there is a mild and distributional solution* $u = u(u_0) \in C([0,T]; L^1(\rho\, dx))$ *with*

$$|u(t)|_\infty \leq \exp\left(|(div_\rho D)^- + |D||_\infty^{\frac{1}{2}} t\right)|u_0|_\infty, \quad \forall t \geq 0.$$

Moreover, $S(t) : u_0 \mapsto u(t)$, *is a semigroup of* $L^1(\rho)$*-contractions, i.e.*

$$S(t+s)u_0 = S(t)S(s)u_0,\ S(0) = Id,$$
$$|S(t)u_0 - S(t)v_0|_{L^1(\rho)} \leq |u_0 - v_0|_{L^1(\rho)}.$$

If $u_0(x)\rho(x)$ *is a probability density, so is each* $u(t)\rho(x)$.

If in addition β is Lipschitz, everything holds for $u_0 \in L^1(\rho)$ instead of $u_0 \in L^1(\rho) \cap L^\infty$.

Definition

(i) Let $A : D(A) \subseteq X \to X$ be a nonlinear operator in a Banach space X. A **mild solution** *to*

$$\frac{d}{dt}u(t) = -A(u(t)), \quad u(0) = u_0, \quad t \geq 0,$$

is $u \in C(\mathbb{R}_+, X)$ such that $u = \lim^X_{h\to 0} u_h$ locally uniformly in time,

$$\begin{aligned} &u_h(0) := u_h^0 := u_0, \\ &u_h(t) := u_h^i, \forall t \in ((i-1)h, ih], i \in \mathbb{N}, \\ &u_h^i \in D(A), u_h^i + hAu_h^i = u_h^{i-1}. \end{aligned}$$

(ii) A **mild solution to** (FP_ρ) *is a mild solution in $X = L^1(\rho dx)$ with*

$$A_0(u) = -\Delta\beta(u) - \nabla\Phi \cdot \nabla\beta(u) - div_\rho\big(D(x)b(u)u\big).$$

Idea of proof: Apply Crandall–Liggett theorem, i.e.

A m–accretive $\Longrightarrow \forall u_0 \in \overline{D(A)}^X$: $\exists$!mild sol. to $\frac{d}{dt}u(t) = -Au(t), u(0) = u_0$, forming a semigroup of $|\cdot|_X$-contractions.

A is *m–accretive*, if

$$R(\mathrm{Id} + \lambda A) = X, \forall \lambda > 0 \text{ and } |(\mathrm{Id} + \lambda A)^{-1} f - (\mathrm{Id} + \lambda A)^{-1} g|_X \leq |f - g|_X.$$

For A_0 as above, one can prove the full-range-condition in $(X, |\cdot|_X) = (L^1(\rho\, dx) \cap L^\infty, |\cdot|_{L^1(\rho)})$, and there exists an m–accretive restriction $A \subseteq A_0$. Then, by small relaxations of the original C.–L.-proof the existence claim follows.

Price to pay: Since range- and contraction property only in $L^1(\rho\, dx) \cap L^\infty$, cannot conclude uniqueness of the mild solution, only uniqueness among solutions approximated by step functions with values in $L^1(\rho\, dx) \cap L^\infty$.

Solving

$$v + \lambda A v = f, \quad f \in L^1(\rho\, dx) \cap L^\infty$$

similar to [BarbuRöckner21]: Approximate A by smooth A_ε, $\varepsilon > 0$, and use

$$\text{symmetry of } Lv := \Delta v - \nabla \Phi \cdot \nabla v \text{ in } L^2(\rho\, dx),$$

i.e. $(L, L^2(\rho\, dx))$ replaces $(\Delta, L^2(dx))$ in [BarbuRöckner21].

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Superposition principle: For any weakly continuous distributional probability density solution $t \mapsto u_t$ to the Nemytskii-type nlFPE

$$du_t = \partial_{ij}^2(a_{ij}(t,u_t(x),x)u_t) - \partial_i(b_i(t,u_t(x),x)u_t), \quad t \geq s, \tag{1}$$

there exists a weak solution X to

$$dX_t = b(t,u_t(X_t),X_t)dt + \sigma(t,u_t(X_t),X_t)dB_t, \quad t \geq s$$

with $\mathscr{L}_{X_t} = u_t(x)dx$.

Note:

$$\partial_t u = \Delta\beta(u) - \nabla\Phi \cdot \nabla\beta(u) - \mathrm{div}_\rho(D(x)b(u)u)$$

appears not of the above type. **But:** due to symmetry of $\Delta - \nabla\Phi$ in $L^2(\rho\, dx)$: Distributional formulation (i.e. integrating with test fct., putting derivatives to test fct.) is of same type as (1).

Hence: Superposition principle applies: For any weakly continuous solution $t \mapsto u(t,x)$ such that $u(t,x)\rho(x)dx \in \mathscr{P}$ to

$$\partial_t u = \Delta\beta(u) - \nabla\Phi \cdot \nabla\beta(u) - \mathrm{div}_\rho(D(x)b(u)u),$$

there exists a weak solution X to

$$dX_t = \left[D(X_t)b(v(t,X_t)\rho^{-1}(X_t)) - \frac{\beta(v(t,X_t)\rho^{-1}(X_t))}{v(t,X_t)\rho^{-1}(X_t)}\nabla\Phi(X_t)\right]dt + \sqrt{2\frac{\beta(v(t,X_t)\rho^{-1}(X_t))}{v(t,X_t)\rho^{-1}(X_t)}}dB_t,$$
$$\mathscr{L}_{X_t} = v(t,x)dx := u(t,x)\rho(x)dx.$$

$\implies$ We can solve this SDE for initial data

$$\mathscr{L}_{X_0} = u(0,x)\rho(x)dx,\ \ u(0,\cdot) \in L^1(\rho\, dx) \cap L^\infty.$$

This SDE is a model for *nonlinear generalized perturbed Ornstein–Uhlenbeck processes*. Indeed: For $\beta = \mathrm{Id}$, $\Phi = \frac{|x|^2}{2}$ and $D = 0$, it reduces to

$$dX_t = -X_t dt + dB_t.$$

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We want to show: The family of solution path laws $\mathbb{P}_{u_0}$, $u_0 \in L^1(\rho\, dx) \cap L^\infty$, to the above SDE with initial datum $u_0 \rho\, dx \in \mathscr{P}$ and one-dim. marginals

$$\mathbb{P}_{u_0} \circ \pi_t^{-1} = S(t)u_0, \quad t \geq 0,$$

is a *nonlinear Markov process*, i.e. satisfies the following definition. Set $\mathscr{F}_r := \sigma(\pi_s, 0 \leq s \leq r) \subseteq \mathscr{B}(C(\mathbb{R}_+, \mathbb{R}^d))$.

Definition

$\{\mathbb{P}_{u_0}, u_0 \in L^1(\rho\, dx) \cap L^\infty\}$ is a *nonlinear Markov process*, if $\forall A \in \mathscr{B}(\mathbb{R}^d)$, $0 \leq r \leq t, u_0 \in L^1(\rho\, dx) \cap L^\infty$

$$\mathbb{P}_{u_0}(\pi_t \in A | \mathscr{F}_r)(\cdot) = \mathbb{P}_{S(r)u_0}(\pi_{t-r} \in A | \pi_0 = \pi_r(\cdot))\ \mathbb{P}_{u_0} - \text{a.s.}$$

We can apply the following result:

Theorem (R./Röckner22)

If $\mathscr{P}_0 \subseteq \mathscr{P}$ and $(S(t))_{t\geq 0}, S(t) : \mathscr{P}_0 \to \mathscr{P}_0$, is a semigroup of distributional solutions to the FPE

$$\partial_t \mu_t = \partial^2_{ij}\big(a_{ij}(t,\mu_t,x)\mu_t\big) - \partial_i\big(b_i(t,\mu_t,x)\mu_t\big),$$

and if $t \mapsto S(t)\mu_0$, $\mu_0 \in \mathscr{P}_0$, is the unique solution to the linearized FPE

$$\partial_t \nu_t = \partial^2_{ij}\big(a_{ij}(t,S(t)\mu_0,x)\nu_t\big) - \partial_i\big(b_i(t,S(t)\mu_0,x)\nu_t\big), \nu_0 = S(0)\mu_0,$$

then the superposition solutions $\{\mathbb{P}_{\mu_0}\}_{\mu_0\in\mathscr{P}_0}$ to the corresponding DDSDE are a nonlinear Markov process with marginals $S(t)\mu_0$, $t \geq 0, \mu_0 \in \mathscr{P}_0$.

We can prove the linearized uniqueness claim, and hence obtain:

The solution path laws $\{\mathbb{P}_{u_0}\}_{u_0\in L^1(\rho dx)\cap L^\infty}$ to the nl-gen-per-OU-equation, obtained as superposition solutions of the semigroup solution to the weighted FPE, are a nonlinear Markov process.

Thank you for your attention!