The p-Laplace operator and p-Brownian motion

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Table of Contents

Heat equation and Brownian motion

Fundamental link between PDE-theory and stochastic analysis: connection between *heat equation*

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \qquad \text{(HE)}$$

and *Brownian motion* $B = (B_t)_{t \geq 0}$, precisely:
The curve of *one-dimensional time marginals* of B

$$t \mapsto \mu_t^0 := \mathscr{L}(B_t), \quad t \geq 0,$$

is the fundamental solution to (HE) with initial value δ_0, i.e. the *heat kernel*

$$p(t,y,0) = (2\pi t)^{-\frac{d}{2}} \exp\left(\frac{-|y|^2}{2t}\right), \quad t > 0,$$

in the sense that

$$\mu_t^0 = p(t,y,0)dy, \quad t > 0, \quad \mu_t^0 \xrightarrow{t \to 0} \delta_0.$$

Similarly, for $x \in \mathbb{R}^d$, marginal curve $(\mu_t^x)_{t\geq 0}$ of Brownian motion from x,

$$B^x = (B_t^x)_{t\geq 0} = (B_t + x)_{t\geq 0},$$

is the fundamental solution to (HE) with initial value δ_x, i.e.

$$\mu_t^x = p(t,y,x)dy = p(t,x-y,0)dy, \quad t > 0.$$

Of course, B^x solves the SDE

$$dX_t^x = dB_t, \quad X_0^x = x, \tag{SDE}$$

and the family of path laws

$$(\mathbb{W}_x)_{x\in\mathbb{R}^d}, \quad \mathbb{W}_x := \mathscr{L}(B^x),$$

is Markov i.e. for $A \in \mathscr{B}(\mathbb{R}^d)$ and $\pi_t : C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d, \pi_t(w) := w(t)$,

$$\mathbb{W}_x(\pi_{t+s} \in A | \mathscr{F}_s)(\cdot) = \mathbb{W}_{\pi_s(\cdot)}(\pi_t \in A), \quad \mathbb{P}_x - \text{a.s.},$$

and $(\mu_t^x)_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d} =$ is transition kernel. ($\mathbb{W}_0 =$ standard Wiener measure)

Conclusion: Fundamental solutions of (HE) are the one-dimensional time marginal densities of a uniquely determined Markov process, which consists of solution path laws $\mathbb{W}^x$ to (SDE).

Table of Contents

p-Laplacian and main question

Instead of (HE), now consider *parabolic p-Laplace equation*

$$\partial_t u(t,x) = \operatorname{div}\left(|\nabla u(t,x)|^{p-2}\nabla u(t,x)\right), \quad (t,x) \in (0,\infty)\times\mathbb{R}^d, \qquad \text{(pL)}$$

$p > 2$. $p = 2$: Heat equation.
Recall: $\exists$ fundamental solution: *Barenblatt solution*

$$w(t,y,x) = t^{-k}\left(C_1 - qt^{-\frac{kp}{d(p-1)}}|x-y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \quad (t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,$$

where $C_1 > 0$ such that $\int_{\mathbb{R}^d} w(t,y,x)dy = 1$ for all (t,x). Since $w(t,y,x)dy \to \delta_x$ weakly as $t \to 0$, write $w(0,y,x)dy := \delta_x(dy)$.

Question: Analogue connection to stochastic analysis as for heat equation and Brownian motion?
Precisely: Is there a naturally associated SDE with unique solutions, which deserve to be called *p-Brownian motion*, and do their path laws form a Markov process, which is uniquely determined by $w(t,y,x)$?
Answer: YES!, this is the contribution of our work.

Table of Contents

Our result is obtained by applying *and extending* a general approach to connect nonlinear PDEs, McKean–Vlasov SDEs and "nonlinear" Markov processes:

Consider the **nonlinear Fokker–Planck equation (NLFPE)**

$$\partial_t \mu_t = \partial^2_{ij}(a(\mu_t,x)\mu_t) - \partial_i(b_i(\mu_t,x)\mu_t), \quad t \geq 0, \quad \mu_0 = \zeta, \qquad \text{(NLFPE)}$$

a 2nd-order parabolic PDE for probability measures $\mu_t \in \mathscr{P}(\mathbb{R}^d)$; notion of solution: distributional, no a priori regularity of a, b and μ_t needed.

Associated SDE is the **McKean–Vlasov equation**

$$dX_t = b(\mathscr{L}(X_t), X_t)dt + \sigma(\mathscr{L}(X_t), X_t)dB_t, \quad t \geq 0, \quad \mathscr{L}(X_0) = \zeta, \qquad \text{(MVSDE)}$$

where $\frac{1}{2}\sigma\sigma^T = a$. Known:

- X weak solution to (MVSDE) $\implies$ $(\mu_t)_{t\geq 0} := (\mathscr{L}(X_t))_{t\geq 0}$ solves (NLFPE).
- $(\mu_t)_{t\geq 0}$ solution to (NLFPE) $\implies$ $\exists$ weak solution X to (MVSDE) with $\mathscr{L}(X_t) = \mu_t$ ("*superposition principle*").

Even if (MVSDE) has unique solutions for all initial data δ_x, the unique solution laws $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ do NOT form a Markov process in the usual sense.

But: They satisfy a suitable "nonlinear Markov property":

$$\mathbb{P}_x(\pi_{s+t} \in A|\mathscr{F}_s)(\cdot) = \mathbb{P}_{\mu_s^x}(\pi_t \in A|\pi_0 = \pi_0(\cdot))\ \mathbb{P}_x - \text{a.s.}, \qquad (n\ell\text{MP})$$

where $\mathbb{P}_{\mu_s^x}$ denotes the solution to (MVSDE) with random initial condition $\mu_s^x := \mathbb{P}_x \circ \pi_s^{-1}$.

Note:

$$\mathbb{P}_{\mu_s^x} \neq \int_{\mathbb{R}^d} \mathbb{P}_y \, d\mu_s^x(dy),$$

so the RHS is NOT equal to $\mathbb{P}_{\pi_s(\cdot)}(\pi_t \in A)$ as in the classical case.

Main result of a previous paper:

Theorem (R./Röckner23)

Let $\mathscr{P}_0 \subseteq \mathscr{P}$ be a class of admissible initial conditions and $(\mu^\zeta)_{\zeta \in \mathscr{P}}$, $\mu^\zeta = (\mu_t^\zeta)_{t \geq 0}$ a family of solutions to the NLFPE such that

(i) $\mu_{t+s}^\zeta = \mu_t^{\mu_s^\zeta}, \quad \forall s,t \geq 0, \zeta \in \mathscr{P}_0$ *("flow property")*

(ii) μ^ζ *is an extreme point in the convex set of solutions to* **linear** *FPE*

$$\partial_t \mu_t = \partial_{ij}^2 \big(a_{ij}(\mu_t^\zeta, x)\mu_t\big) - \partial_i \big(b_i(\mu_t^\zeta, x)\mu_t\big), \quad t \geq 0, \quad \mu_t \xrightarrow{t \to 0} \zeta$$

for all ζ from a sufficiently large class $\mathfrak{P}_0 \subseteq \mathscr{P}_0$.

Then, for all $\zeta \in \mathscr{P}_0$ there is a weak solution $X^\zeta = (X_t^\zeta)_{t \geq 0}$ to the MVSDE such that

(a) $\mathscr{L}(X_t^\zeta) = \mu_t^\zeta, \quad t \geq 0,$

(b) X^ζ *is the unique (!) solution to MVSDE with (a),*

(c) $(\mathbb{P}_\zeta)_{\zeta \in \mathscr{P}_0}$, $\mathbb{P}_\zeta := \mathscr{L}(X^\zeta)$, *is a nonlinear Markov process.*

In words: **Given** solution family $(\mu^\zeta)_{\zeta\in\mathscr{P}_0}$ for NLFPE satisfying (i)+(ii) we **construct** a nonlinear Markov process, which

(I) has one-dimensional time marginals μ_t^ζ, $t\geq 0$, $\zeta\in\mathscr{P}_0$,

(II) consists of weak solutions to the associated McKean–Vlasov SDE,

(III) is uniquely determined by $(\mu^\zeta)_{\zeta\in\mathscr{P}_0}$ and the MVSDE.

Table of Contents

To treat nonlinear PDEs as NLFPEs, study *Nemytskii-case*:

$$a_{ij}(\mu,x)=\tilde{a}_{ij}(u(x),x), \quad b_i(\mu,x)=\tilde{b}(u(x),x),$$

where $\mu=u(x)dx$, i.e. *pointwise density dependence in measure variable.*

Rewriting NLFPE as equation for u yields the PDE

$$\partial_t u(t,x)=\partial_{ij}^2(a(u(t,x),x)u(t,x))-\partial_i(b_i(u(t,x),x)u(t,x)), \quad (n\ell\text{PDE})$$
$$u(t,x)dx \xrightarrow{t\to 0} \zeta,$$

with associated McKean–Vlasov SDE

$$dX_t=b(u(t,X_t),X_t)dt+\sigma(u(t,X_t),X_t)dB_t,$$
$$\mathscr{L}(X_t)=u(t,x)dx, \quad t>0.$$

Such coefficients are **not** continuous in their measure variable. **But** this is *not* required in the previous result: We only have to check (i)+(ii). By PDE-methods, we did this for: PME, generalized PME, Burgers, $2D$ vorticity EE and NSE, equations with fractional Laplace,...

Table of Contents

The p-Laplace equation

$$\partial_t u(t,x) = \operatorname{div}\left(|\nabla u(t,x)|^{p-2}\nabla u(t,x)\right)$$

seems not to be of type

$$\partial_t \mu_t = \partial^2_{ij}(a(\mu_t,x)\mu_t) - \partial_i(b_i(\mu_t,x)\mu_t).$$

But in fact we **1)** rewrite it *formally* as

$$\partial_t u(t,x) = \Delta\left(|\nabla u(t,x)|^{p-2}u(t,x)\right) - \operatorname{div}\left(\nabla(|\nabla u(t,x)|^{p-2})u(t,x)\right), \tag{pL-FPE}$$

and **2)** show *rigorously*: A weak solution $u \in L^{p-1}_{\text{loc}}((0,\infty);W^{1,p-1}_{\text{loc}})$ to the p-Laplace equation is also a weak solution to (pL-FPE) if and only if all appearing gradient-terms are defined.

Note: (pL-FPE) is a nonlinear FPE with coefficients

$$a_{ij}(u,x) = \delta_{ij}|\nabla u(x)|^{p-2}, \quad b_i(u,x) = \partial_i\left(|\nabla u(x)|^{p-2}\right),$$

i.e. coefficients depend on their measure argument $\mu = u(x)dx$ pointwise via first- and second-order derivatives of density u (new!).

The (at first: formally) associated McKean–Vlasov SDE is

$$\begin{aligned} dX(t) &= \nabla(|\nabla u(t,X(t))|^{p-2})dt + |\nabla u(t,X(t))|^{\frac{p-2}{2}} dW(t), \\ \mathscr{L}(X(t)) &= u(t,x)dx,\ t>0. \end{aligned}$$

Our first result is the following probabilistic representation of solutions u to the p-Laplace equation in terms of the above McKean–Vlasov SDE.

Proposition (Barbu, R., Röckner24)

Let u be a weak solution to the p-Laplace equation such that

$$|\nabla u|^{p-2} \in L^1_{\text{loc}}((0,\infty); W^{1,1}_{\text{loc}}(\mathbb{R}^d)).$$

and

$$\int_0^T \int_{\mathbb{R}^d} \left(|\nabla u|^{p-2} + |\nabla(|\nabla u|^{p-2})|\right) u \, dxdt < \infty, \quad \forall T > 0.$$

Then there exists a weak solution $X = (X_t)_{t\geq 0}$ to the above McKean–Vlasov SDE such that $\mathscr{L}(X_t) = u(t,x)dx$ for all $t \geq 0$.

Table of Contents

Application to Barenblatt solution and main result

We show: The previous proposition applies to the Barenblatt solution

$$(t,y) \mapsto w(t,y,x) = t^{-k}\left(C_1 - qt^{-\frac{kp}{d(p-1)}}|x-y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}},$$

where $x \in \mathbb{R}^d$ denotes the initial value, i.e. $w(t,y,x)dy \xrightarrow{t\to 0} \delta_x$. Hence:

$$\exists X^x = (X_t^x)_{t\geq 0} \text{ solution to McKean–Vlasov SDE with}$$

$$\mathscr{L}(X_t^x) = w(t,y,x)dy, \quad t > 0.$$

But we prove much more:

Theorem (Barbu, R., Röckner24, Main Result)

Let $d \geq 2$, $p > 2(1+\frac{1}{d})$, *and* $\mathscr{P}_0 = \{w(\delta, y, x)dy, x \in \mathbb{R}^d, \delta \geq 0\}$. *For each* $\zeta = w(\delta, y, x)dy \in \mathscr{P}_0$, *there is a unique weak solution* $X^\zeta = (X^\zeta)_{t\geq 0}$ *to the McKean–Vlasov SDE such that*

$$\mathscr{L}(X_t^\zeta) = w(\delta + t, y, x)dy, \quad \forall t \geq 0$$

and the solution laws $(\mathbb{P}_\zeta)_{\zeta \in \mathscr{P}_0}$, $\mathbb{P}_\zeta := \mathscr{L}(X^\zeta)$, *form a nonlinear Markov process. In particular,* X^x *from the previous page is unique. Moreover, this nonlinear Markov process is uniquely determined by the McKean–Vlasov SDE and its time marginals* $w(t, y, x)dy$, $t \geq 0, x \in \mathbb{R}^d$.

We also show that $(\mathbb{P}_\zeta)_{\zeta \in \mathscr{P}_0}$ is uniquely determined by $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$, so that "$(\mathbb{P}_\zeta)_{\zeta \in \mathscr{P}_0} = (\mathbb{P}_x)_{x\in\mathbb{R}^d}$".

Definition (Barbu, R., Röckner24)

We call the nonlinear Markov process $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ *p-Brownian motion*, and $\mathbb{P}_x$ (equivalently: X^x) *p-Brownian motion started at* x.

Comparison with classical case $p = 2$

$p = 2$: Fundamental solution $p(t, y, x)dy$ to **heat equation** determine a unique Markov process $(\mathbb{W}_x)_{x\in\mathbb{R}^d}$, consisting of unique solution laws to associated SDE

$$dX_t^x = dB_t, \quad X_0^x = x,$$

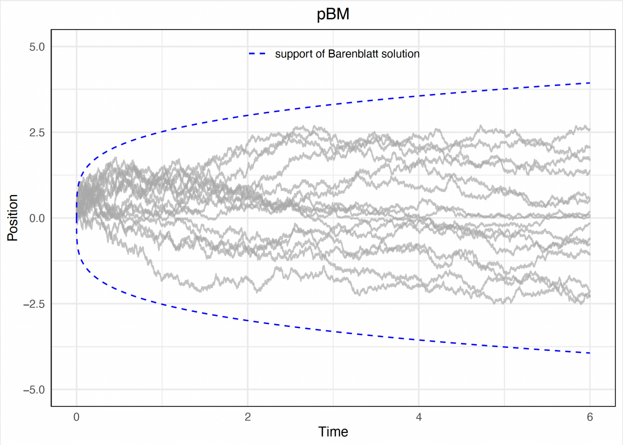
and X^x is Brownian motion started at x.

$p > 2$: Fundamental solutions $w(t, y, x)dy$ to **p-Laplace equation** determine a unique nonlinear Markov process $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$, consisting of unique solution laws to associatd McKean–Vlasov SDE

$$\begin{aligned} dX_t^x &= \nabla(|\nabla u(t, X_t^x)|^{p-2})dt + |\nabla u(t, X_t^x)|^{\frac{p-2}{2}} dW(t), \\ \mathscr{L}(X_t^x) &= u(t, y)dy = w(t, y, x)dx, \ t > 0. \end{aligned}$$

Paths of Brownian motion and p-Brownian motion

Simulation of p-Brownian motion paths for $p = 3$:



Credit to Ehsan Abedi, Doctoral student @UBielefeld.

Idea of proof

Idea: Apply theorem from [R./Röckner23]:

Choose $\mathscr{P}_0 = \{w(\delta, y, x)dy,\ \delta \geq 0, x \in \mathbb{R}^d\}$. For initial datum $\zeta = w(\delta, y, x)dy$, have the solution $\mu^\zeta = (w(t, +\delta, y, x)dy)_{t\geq 0}$. $\{\mu^\zeta\}_{\zeta \in \mathscr{P}_0}$ has flow property ✓.

Remains to show: $\exists$ sufficiently large set $\mathfrak{P}_0 \subseteq \mathscr{P}_0$ such that for $\zeta = w(\delta, y, x)dy \in \mathfrak{P}_0$, $\mu^\zeta = (w(\delta + t, y, x)dy)_{t\geq 0}$ is extreme point in the set of solutions to the linear FPE

$$\partial_t \mu_t = \Delta\big(|\nabla w(\delta + t, y, x)|^{p-2}\mu_t\big) - \operatorname{div}\big(\nabla(|\nabla w(\delta + t, y, x)|^{p-2})\mu_t)\big)$$
$$\mu_0 = w(\delta, y, x)dy$$

We can prove this for all solutions μ^ζ with

$$\zeta \in \mathfrak{P}_0 := \{w(\delta, y, x)dy, x \in \mathbb{R}^d,\ \delta > \mathbf{0}\},$$

and we show that $\mathfrak{P}_0$ is sufficiently large.
How do we prove this geometric property?

We can characterize this extremality condition as follows:

Lemma (R./Röckner23)

For $\zeta = w(\delta, y, x)dy$, $t \mapsto \mu^\zeta$ *is an extreme point in the set of solutions to*

$$\begin{aligned} \partial_t \mu_t &= \Delta\big(|\nabla w(\delta + t, y, x)|^{p-2}\mu_t\big) - \operatorname{div}\big(\nabla(|\nabla w(\delta + t, y, x)|^{p-2})\mu_t)\big) \\ \mu_0 &= w(\delta, y, x)dy \end{aligned}$$

iff $t \mapsto \mu_t^\zeta$ *is the only solution to this equation in the class*

$$\mathscr{A}_{\delta,x} := \big\{\mu \in C_{\mathrm{w}}([0,\infty); \mathscr{P}) : \mu_t \leq C\mu_t^\zeta, \quad \forall t \geq 0 \text{ for some } C > 0\big\}.$$

Note: $\mathscr{A}_{\delta,x}$ consists of measures with densities w.r.t. Lebesgue measure, so the above FPE is a PDE.
$\implies$ Have to prove a restricted uniqueness result for a class of very degenerate linear PDEs. This was hard, but using precisely the definition and properties of $w(t, y, x)$, it worked for $\delta > 0$! □

Table of Contents

Future work

(i) Path properties of p-Brownian motion

(ii) What happens for $p \to 2$?

(iii) SDEs driven by p-Brownian motion

(v) Nonlinear generator and semigroup for the associated nonlinear Markov process

(vi) Similar program for related equations (e.g. Leibenson equation on manifolds)

(vii) ...many more...there is a lot to do!

Thank you for your attention!