The Leibenson equation as a nonlinear Fokker-Planck equation and its associated nonlinear Markov process

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Motivation: Heat equation and Brownian motion

Fundamental link of PDE-theory and stochastic analysis: connection between *heat equation*

$$\partial_t u(t,y) = \frac{1}{2}\Delta u(t,y), \quad (t,y) \in (0,\infty)\times\mathbb{R}^d \tag{HE}$$

and d-dim. *Brownian motion* $B = (B_t)_{t\geq 0}$: $1D$*-time marginal curve* of B

$$t \mapsto \mu_t^0 := \mathscr{L}(B_t), \quad t \geq 0,$$

is the fundamental solution to (HE) with initial datum δ_0, i.e. the *heat kernel*

$$p(t,y,0) = (2\pi t)^{-\frac{d}{2}} \exp\left(\frac{-|y|^2}{2t}\right), \quad t > 0,$$

precisely:

$$\mu_t^0 = p(t,y,0)dy, \quad t > 0, \quad \mu_t^0 \xrightarrow{t\to 0} \delta_0.$$

Motivation: Heat equation and Brownian motion

Similarly, for $x \in \mathbb{R}^d$ marginal curve $(\mu_t^x)_{t\geq 0}$ of Brownian motion from x

$$B^x = (B_t^x)_{t\geq 0} := (B_t + x)_{t\geq 0},$$

is the fundamental solution to (HE) with initial datum δ_x, i.e.

$$\mu_t^x = p(t,y,x)dy := p(t,x-y,0)dy, \quad t > 0.$$

Of course, B^x solves the SDE

$$dX_t = dB_t, \quad X_0 = x, \tag{SDE}$$

and the family of path laws $(\mathbb{W}_x)_{x\in\mathbb{R}^d}, \mathbb{W}_x := \mathscr{L}(B^x)$, is a Markov process: For $A \in \mathscr{B}(\mathbb{R}^d)$ and $\pi_t : C(\mathbb{R}_+,\mathbb{R}^d) \to \mathbb{R}^d, \pi_t(w) := w(t)$, *Markov property*

$$\mathbb{W}_x(\pi_{t+s} \in A|\mathscr{F}_s)(\cdot) = \mathbb{W}_{\pi_s(\cdot)}(\pi_t \in A), \quad \mathbb{W}_x - \text{a.s.},$$

holds, with transition kernels $(x,A) \mapsto \mu_t^x(A)$, $t \geq 0$.

Conclusion: Fundamental solutions of (HE) are time marginal densities of a uniquely determined Markov process, consisting of solutions to (SDE).

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The Leibenson equation

Instead of (HE), now consider the doubly nonlinear *Leibenson equation*

$$\partial_t u(t,y) = \Delta_p(u^q(t,y)), \quad q > 0, p > 1, \quad (t,y) \in (0,\infty) \times \mathbb{R}^d, \qquad \text{(L)}$$

where $\Delta_p : f \mapsto \operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right)$ p-Laplace operator. Note:

- $q = 1$: p-Laplace equation
- $p = 2$: $\Delta_2 = \Delta$, Porous Media equation
- $(p,q) = (2,1)$: Heat equation (linear).

$\exists$ explicit *Barenblatt solutions*, for $\mathbf{q(p-1) > 1}$:

$$w^x(t,y) := t^{-\frac{d}{\beta}} \left[C - \kappa\left(t^{-\frac{1}{\beta}}|x-y|\right)^{\frac{p}{p-1}} \right]_+^{\gamma}, \quad (t,y) \in (0,\infty) \times \mathbb{R}^d.$$

$\gamma, \beta, \kappa > 0$ depend on p, q, d. $C > 0$ chosen s.t. $\int_{\mathbb{R}^d} w^x(t,y)dy = 1$ for all $t > 0, x \in \mathbb{R}^d$. Initial datum: $w^x(t,y)dy \xrightarrow{t \to 0} \delta_x$ weakly.

Our aim: Establish connection of (L) to an associated SDE and Markov process, as for heat equation and Brownian motion.

Consider the McKean–Vlasov SDE

$$\begin{cases} \mathrm{d}X_t & = \nabla\big(|\nabla u(t,X_t)|^{p-2}u(t,X_t)^{(q-1)(p-1)}\big)\mathrm{d}t \\ & \quad +\big(|\nabla u(t,X_t)|^{\frac{p-2}{2}}u(t,X_t)^{\frac{(q-1)(p-1)}{2}}\big)\mathrm{d}B_t, \\ \mathscr{L}(X_t) = & u(t,x)\mathrm{d}x. \end{cases} \tag{L-SDE}$$

This turns out to be "the correct stochastic analogue" of the Leibenson equation. Note:

(i) Coefficients irregular (BV-drift), degenerate ($u = w^x$ comp. supp.), singular ($(q-1)(p-1)$ may be < 0).

(ii) Drift $= \nabla\sqrt{\text{diffusion}}$.

Theorem (Barbu/Grube/R./Röckner25)

Let $d \geq 2$, $p > \frac{d}{d-1}$ with $q(p-1) > \max\left(1, \frac{2-p+d}{d}\right)$.

There exists a family of path laws $\{\mathbb{L}^x\}_{x\in\mathbb{R}^d}$ such that

(i) $\mathbb{L}^x$ is the path law of the unique (!) weak solution X^x to (L-SDE) *with time marginals $\mathscr{L}(X_t^x) = w^x(t,y)dy$.*

(ii) $\{\mathbb{L}^x\}_{x\in\mathbb{R}^d}$ is a nonlinear Markov process (explained below).

We call $\{\mathbb{L}^x\}_{x\in\mathbb{R}^d}$ *Leibenson process.*

Comparison to classical and recent special cases

$q=1, p=2$ (classical, linear): Fundamental solutions $p(t,y,x)$ to **heat equation** determine a unique Markov process $(\mathbb{W}_x)_{x\in\mathbb{R}^d}$ called *Brownian motion*, consisting of unique solution laws to associated SDE

$$dX_t^x = dB_t, \quad X_0^x = x.$$

$q=1, p>2$ (previously in [Barbu/R./Röckner24]): Fundamental solutions $w^x(t,y)$ to **p-Laplace equation** determine a unique nonlinear Markov process $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ called *p-Brownian motion*, consisting of unique solution laws to associated MV-SDE

$$\begin{aligned} dX_t^x &= \nabla(|\nabla u(t,X_t^x)|^{p-2})dt + |\nabla u(t,X_t^x)|^{\frac{p-2}{2}} dB_t, \\ \mathscr{L}(X_t^x) &= u(t,y)dy = w^x(t,y)dx, \ t\geq 0. \end{aligned}$$

$q(p-1)>1$ (new): Fundamental solutions $w^x(t,y)$ to **Leibenson equation** determine a unique nonlinear Markov process $(\mathbb{L}_x)_{x\in\mathbb{R}^d}$ called *Leibenson process*, consisting of unique solution laws to associated MV-SDE (L-SDE).

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We use *and extend* a method connecting nonlinear PDEs, McKean–Vlasov SDEs and nonlinear Markov processes:

Consider the **nonlinear Fokker–Planck equation (NLFPE)**

$$\partial_t \mu_t = \partial^2_{ij}(a(\mu_t, x)\mu_t) - \partial_i(b_i(\mu_t, x)\mu_t), \quad t \geq 0, \quad \mu_0 = \zeta, \qquad \text{(NLFPE)}$$

a 2nd-order parabolic PDE for probability measures $\mu_t \in \mathscr{P}(\mathbb{R}^d)$; notion of solution: distributional, no a priori regularity of a, b and μ_t needed.

Associated SDE is the **distribution-dependent McKean–Vlasov SDE**

$$dX_t = b(\mathscr{L}(X_t), X_t)dt + \sigma(\mathscr{L}(X_t), X_t)dB_t, \quad t \geq 0, \quad \mathscr{L}(X_0) = \zeta, \qquad \text{(MVSDE)}$$

where $\frac{1}{2}\sigma\sigma^T = a$.

- By Itô-formula, straightforward: X weak solution to (MVSDE) $\implies$ $(\mu_t)_{t\geq 0} := (\mathscr{L}(X_t))_{t\geq 0}$ solves (NLFPE).
- $(\mu_t)_{t\geq 0}$ solution to (NLFPE) $\implies$ $\exists$ weak solution X to (MVSDE) with $\mathscr{L}(X_t) = \mu_t$ ("*superposition principle*"). **Crucial: no** regularity assumptions on σ, b!

To treat nonlinear PDEs as NLFPEs, study *generalized Nemytskii-case*:

$$a_{ij}(\mu,y)=\tilde{a}_{ij}((\Gamma_1 u)(y),y),\quad b_i(\mu,y)=\tilde{b}_i((\Gamma_2 u)(y),y),$$

where $\mu=u(y)dy$ and Γ_i maps u to a function $\Gamma_i u$, typically a differential operator. Classical Neymtskii-case: $\Gamma_1 u=\Gamma_2 u=u$.

Rewriting the NLFPE as equation for densities u yields the nonlinear PDE

$$\partial_t u(t,y)=\partial^2_{ij}(\tilde{a}_{ij}((\Gamma_1 u(t))(y),y)u(t,y))-\partial_i(\tilde{b}_i((\Gamma_2 u(t))(y),y)u(t,y)), \tag{nlPDE}$$

$$u(t,y)dy\xrightarrow{t\to 0}\zeta,$$

with associated McKean–Vlasov SDE

$$\begin{aligned}dX_t&=\tilde{b}((\Gamma_2 u(t))(X_t),X_t)dt+\tilde{\sigma}((\Gamma_2 u(t))(X_t),X_t)dB_t,\\ \mathscr{L}(X_t)&=u(t,y)dy,\quad t\geq 0.\end{aligned}$$

Such Nemytskii-coefficients are **not** continuous in their measure variable.

MVSDE-solutions are not Markov

Even if (MVSDE) has unique solutions for all initial data δ_x, the unique solution laws $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ do NOT form a Markov process in the usual sense.

But: They satisfy a suitable "nonlinear Markov property":

$$\mathbb{P}_x(\pi_{s+t} \in A|\mathscr{F}_s)(\cdot) = \mathbb{P}_{\mu_s^x}(\pi_t \in A|\pi_0 = \pi_s(\cdot))\ \mathbb{P}_x - \text{a.s.}, \qquad (nlMP)$$

where $\mathbb{P}_{\mu_s^x}$ denotes the solution to (MVSDE) with initial distribution $\mu_s^x := \mathbb{P}_x \circ \pi_s^{-1}$.

Since

$$\mathbb{P}_{\mu_s^x} \neq \int_{\mathbb{R}^d} \mathbb{P}_y d\mu_s^x(dy),$$

the RHS of (nlMP) is NOT equal to $\mathbb{P}_{\pi_s(\cdot)}(\pi_t \in A)$ as in the classical case.

Path law families $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ satisfying (nlMP) are called *nonlinear Markov processes*.

The main tool for the proof of our result is the following construction of nonlinear Markov processes from [R./Röckner25].

For $s > 0$ and a measure-valued curve $\mu = (\mu_t)_{t\geq 0}$, write $\mu_{s+} := (\mu_{s+t})_{t\geq 0}$.

Theorem (R./Röckner25)

Let $\{\mu^x\}_{x\in\mathbb{R}^d}$, $\mu^x = (\mu^x_t)_{t\geq 0}$, be a family of solutions to a NLFPE with $\mu^x_0 = \delta_x$ such that
(i) $(t,x) \mapsto \mu^x_t$ is injective.
(ii) for any $s > 0$ and $x \in \mathbb{R}^d$, μ^x_{s+} is an extreme point in the set of solutions to the linear FPE

$$\partial_t \nu_t = \partial^2_{ij}\big(a_{ij}(\mu^x_{s+t}, x)\nu_t\big) - \partial_i\big(b_i(\mu^x_{s+t}, x)\nu_t\big), \quad \nu_0 = \mu^x_s.$$

Then there is a nonlinear Markov process $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$, with $\mathbb{P}_x =$ unique solution law to the corresponding MVSDE with $1D$-time marginals μ^x_t.

[R./RöcknerR25], [R./Romito25], [Barbu/Röckner/Zhang25] constructed nonlinear Markov processes associated with (generalized) porous media equation, Burgers, $2D$ vorticity EE and NSE, equations with fractional Laplace, etc.

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First, we identify the Leibenson equation as the NLFPE

$$\partial_t u(t,y) = \Delta\big(|\nabla u(t,y)|^{p-2} u(t,y)^{(p-1)(q-1)} u(t,y)\big) \\ - \operatorname{div}\big(\nabla\big(|\nabla u(t,y)|^{p-2} u(t,y)^{(p-1)(q-1)}\big) u(t,y)\big),$$

i.e. as a generalized Nemytskii-type NLFPE with coefficients

$$a_{ij}(\mu,y) = \delta_{ij} |\nabla u(t,y)|^{p-2} u(t,y)^{(p-1)(q-1)}$$

and

$$b_i(\mu,y) = \partial_i\big(|\nabla u(t,y)|^{p-2} u(t,y)^{(p-1)(q-1)}\big),$$

for $\mu = u(y)dy$. Associated McKean–Vlasov SDE for this NLFPE is

$$\begin{cases} \mathrm{d}X_t &= \nabla\big(|\nabla u(t,X_t)|^{p-2} u(t,X_t)^{(q-1)(p-1)}\big)\mathrm{d}t \\ & \quad +\big(|\nabla u(t,X_t)|^{\frac{p-2}{2}} u(t,X_t)^{\frac{(q-1)(p-1)}{2}}\big)\mathrm{d}B_t, \\ \mathscr{L}(X_t) &= u(t,x)\mathrm{d}x. \end{cases} \tag{L-SDE}$$

Second, we prove that the construction theorem from [R./Röckner25] applies to the Barenblatt solutions $\{w^x\}_{x\in\mathbb{R}^d}$.

Proof of Condition (ii) for Leibenson process

We have to prove: $w^x_{s+} = (w^x(s+t,y)dy)_{t\geq 0}$ is an extreme point in the set of solutions to the linear FPE

$$\partial v_t = \Delta\left(|\nabla w^x(s+t,y)|^{p-2} w^x(s+t,y)^{(p-1)(q-1)} v_t\right) - \operatorname{div}\left(\nabla\left(|\nabla w^x(s+t,y)|^{p-2} w^x(s+t,y)^{(p-1)(q-1)}\right) v_t\right), v_0 = w^x(s,y)dy.$$

The following lemma recasts this condition as a uniqueness condition:

Lemma

The above extremality condition is equivalent to the following restricted uniqueness condition:

w^x_{s+} *is the unique solution to the above linear FPE in the class*

$$\left\{v = (v_t)_{t\geq 0} : v_t \leq C w^x_{s+t}, \forall t \geq 0 \text{ for some } C \geq 1 \text{ independent from } t\right\}$$

This is a uniqueness-problem for a linear PDE with irregular, degenerate, singular coefficients. Proof: hard!, carefully using properties of w^x.

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Thank you for your attention!