Winter term 2026/2027
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Exercises to Fokker–Planck equations

Sheet 3
Class: 20.11.2026

Problem 1 (Weak continuity of marginal curve of SDE solution)**.**

Prove: The one-dimensional marginal curve $t \mapsto \mu_t := \mathcal{L}_{X_t}$ of any weak solution $(X_t)_{t\geqslant 0}$ to the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geqslant 0,$$

is weakly continuous.

Problem 2 (Uniqueness for martingale problem solutions: from Dirac to general initial measures)**.**

Prove Proposition 1.3.11.

Problem 3 (Continuous representatives of FPE solutions)**.**

Let $\mu = (\mu_t)_{t\geqslant 0}$ be a solution to the FPE $\partial_t \mu_t = L^*_{a,b}\mu_t$ with initial condition $\nu \in \mathcal{P}$ such that $a_{ij}, b_i \in L^1_{\text{loc}}([0,\infty) \times \mathbb{R}^d; \mu_t dt)$, such that each μ_t is a nonnegative finite Borel measure und $\operatorname{ess\,sup}_{t\geqslant 0} \mu_t(\mathbb{R}^d) < \infty$ ($\operatorname{ess\,sup}_{t\geqslant 0} \mu_t(\mathbb{R}^d)$ denotes the infimum of numbers $c > 0$ such that $\mu_t(\mathbb{R}^d) \leqslant c$ for all but dt-negligible many $t \geqslant 0$).

Prove: There exists a unique vaguely continuous dt-version $(\tilde{\mu}_t)_{t\geqslant 0}$ (i.e. $\mu_t = \tilde{\mu}_t$ dt-a.s.), and $\tilde{\mu}$ also solves the FPE with initial datum ν.

Hint: It can be a good idea to start from Definition 1.2.1. and with test functions of product type $\varphi(t,x) = f(t)\psi(x)$ to show that $t \mapsto \int \psi \, d\mu_t$ belongs to $W^{1,1}_{\text{loc}}([0,\infty))$. On a one-dimensional space, $W^{1,1}$-functions are quite regular. At some point, you might want to use that every nonnegative continuous linear functional $F : C_0(\mathbb{R}^d) \to \mathbb{R}$ is of type $F(\varphi) = \int \varphi d\mu$ for a finite nonnegative Borel measure μ on $\mathbb{R}^d$, with $\mu(\mathbb{R}^d) = |F|$, where $|\cdot|$ denotes the usual norm on $(C_0)^*$.